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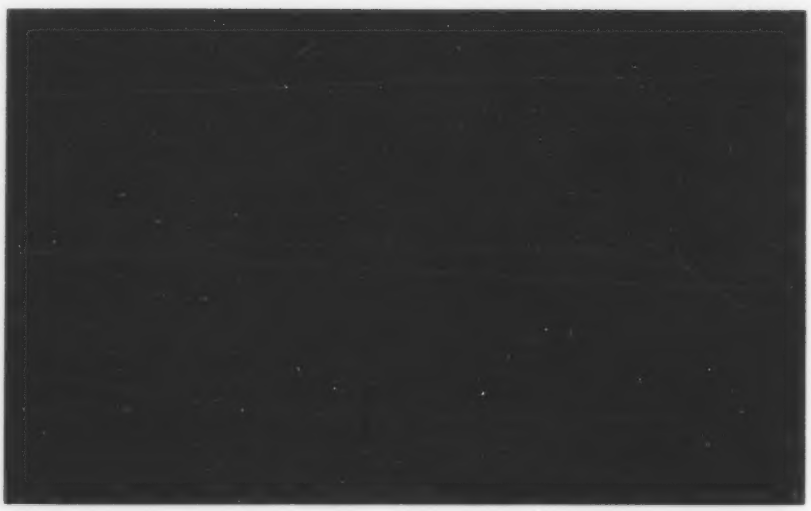

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A New Dynamic Traffic Assignment Model

by

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ABSTRACT

We propose a new Dynamic Traffic Assignment model for the analysis of the variations over time of the traffic flows on a road or street network during a day or peak period. This is a discrete-time model which can handle general networks with multiple origin-destination pairs and various preference patterns with regards to departure and arrival times.

The main element of the model is a "within-day" submodel which performs a macroscopic discrete-time simulation of traffic flows over the network on a specific day for a given set of *a priori* travel decisions by the drivers. This submodel enforces the "first-in first-out" discipline on all links at all times and allows for a detailed modelling of the interactions at road or street junctions.

Day-to-day adjustments in travel decisions are handled by a simple, generic mechanism that leads to a Dynamic Traffic Equilibrium condition, which, under mild assumptions, is shown to be satisfied by at least one traffic pattern.

Key words: dynamic traffic assignment; dynamic traffic equilibrium; discrete-time traffic assignment.

RÉSUMÉ

Nous présentons un nouveau modèle d'affectation dynamique pour l'analyse des variations temporelles des flux de trafic sur un réseau routier ou urbain pendant une journée ou une période de pointe. C'est un modèle en temps discret qui permet le traitement de réseaux généraux avec plusieurs paires origine-destination et divers profils de préférences en ce qui a trait aux heures de départ et d'arrivée.

L'élément central du modèle est un sous-modèle "quotidien" qui réalise une simulation déterministe macroscopique des flux de trafic pour une journée donnée en fonction d'un ensemble de décisions *a priori* de la part des automobilistes en ce qui concerne leur déplacement. Ce sous-modèle permet une modélisation détaillée des interactions aux intersections et maintient l'ordonnancement temporel des véhicules sur les arcs du réseau (règle "premier arrivé-premier parti").

Un second sous-modèle décrit de façon simple et générale le processus par lequel les automobilistes modifient d'une journée à l'autre les décisions relatives à leur déplacement. On montre que ce processus induit une condition d'équilibre dynamique du trafic, et que, sous des hypothèses faibles, il existe toujours au moins un ensemble de flux satisfaisant cette condition.

Mots-clés: affectation dynamique du trafic; équilibre dynamique du trafic; affectation du trafic en temps discret.

1 INTRODUCTION

This paper deals with the modelling of the traffic flows on urban transportation networks, mainly during the peak periods of a day. Most of the research done in the past has concentrated on the Static Traffic Assignment Problem (STAP) and its variants, where a *static* demand is assigned on the network based on the descriptive assignment rules of Wardrop (1952) (see, among others, Dafermos and Sparrow, 1969). Although the demand is not constant over the day time in the urban context, the static models have been extensively used in practice.

During the two past decades, research has been more and more focussed on developing new tools regarding the Dynamic Traffic Assignment Problem (DTAP), that may help to analyze the traffic flow variations over the time, in a city transportation network. The early DTAP models were often based on heuristic concepts (Yagar, 1970; Leonard et al., 1978) where the traffic flows are assigned on minimum cost paths. Other models were too restrictive: some do not apply to capacity constrained links (Robillard, 1974) and others consider only simple networks with only one destination (Hendrickson and Kocur, 1981) or having only parallel paths between the (only) origin-destination pair (Arnott et al., 1990). Finally, some impose some simplifying conditions on the users' knowledge of the traffic condition over the time (Drissi, 1990).

In 1978, Merchant and Nemhauser present a first (discrete-time nonconvex) minimization model for the single origin-destination DTAP. Ho (1980) proposes a stepwise linear version of this model which is more deeply investigated by Carey (1986 and 1987); this author presents a convex version of the model and some techniques to generalize it to the many origins and many destinations case. Finally, Friesz et al. (1989) describe a continuous time version of the model (optimal control problem).

In this paper, we propose a new discrete-time DTAP model for general networks with multiple origin-destination pairs and various preference patterns with regards to departure

and arrival times. This model is not based on an explicit optimization formulation, but rather on an iterative scheme that attempts to capture the learning process which takes place when drivers travel repeatedly on a network over a period covering several days (or identical peak periods). This iterative scheme is made up of two elements: a "within-day model" which describes traffic flows resulting from a set of *a priori* travel decisions on the part of the drivers on a given day, and a "day-to-day" learning model which allows drivers to modify their travel decisions on the basis of the travel times observed on the previous day.

The remainder of the paper is organized as follows. Section 2 is devoted to the description of a restricted version of the "within-day" model in which departure times are fixed. We then indicate in section 3 how departure time decisions, along with all relevant parameters, can be incorporated into this model. In section 4, a simple, generic "day-to-day" decision adjustment mechanism is proposed and is shown to lead to a Dynamic Traffic Equilibrium condition, which, under mild assumptions, is always satisfied by at least one traffic pattern. Future research directions and open questions are discussed in section 5 which concludes the paper.

2 WITHIN DAY DYNAMIC TRAFFIC ASSIGNMENT

In this section, we assume that departure times for all vehicles are fixed and that drivers have chosen *before their departure* a "strategy" for reaching their destination. Our objective is thus to determine the traffic flows and O-D travel times that will result from the simultaneous application of the strategies of all drivers given the congestion phenomena occurring over the network. The methodology we use can be viewed as a macroscopic deterministic discrete-time simulation of traffic flows in which we simply "push" vehicles from their origin towards their destination on a time-space representation of the network according to simple rules.

2.1 Basic Notation and Assumptions

The road network is described by a directed graph $G = (N, A)$ where the node-set N represents road junctions, origins and destinations, and the link-set A represents road or street sections. The time horizon covered by the study is divided into T equal-length time periods. Throughout the paper, it is assumed that T is large enough to allow all vehicles to reach their destination before or in period T.

We also define the following notation:

P, the set of origins,

Q, the set of destinations,

g^t_{pq}, the number of vehicles leaving origin p for destination q during period t, $(p \in P,\ q \in Q,\ t \in \{1, \ldots, T\})$,

A^+_i, the "forward-star" of node i, i.e., $\{(i,j) \in A \text{ for some } j \in N\}, \quad (i \in N)$,

A^-_i, the "backward-star" of node i, i.e., $\{(j,i) \in A \text{ for some } j \in N\}, \quad (i \in N)$,

j_a, the "head-node" of link a, i.e., $a = (i, j_a)$ for some $i \in N, \quad (a \in A)$.

2.2 Modelling Drivers' Travel Strategies

The simplest way to model *a priori* travel strategies is simply to assume that drivers chose a *path* from their origin to their destination. In our opinion, such an approach is unpractical for large networks since it leads naturally to link-path formulations which tend to be unwieldy in a dynamic setting. Furthermore, a priori path choices do not allow for any driver reaction to network conditions.

Instead, we propose to describe travel strategies by indicating for each node of network and time period *the next link to be entered* if one is traveling towards a given destination. This allows for a limited reaction from the drivers to the time-varying flows over the network, while maintaining the advantages of node-link formulations.

In congested networks, it is well known that it is not usually optimal for all vehicles to follow the same route to their destination and that some "flow-splitting" may occur. To account for this possibility, we must allow drivers to distribute themselves among the links coming out of a node. This leads us to the following description of drivers' strategies:

r_{iq}^{ta}: the proportion of vehicles headed towards destination q passing through node i at time t that next enter link a, $\quad (i \in N,\ q \in Q,\ t \in \{1,\ldots,T\},\ a \in A_i^+)$,

with $r_{iq}^{ta} \geq 0$ and $\sum_{a \in A_i^+} r_{iq}^{ta} = 1$, for $i \neq q$ (in the case $i = q$, we simply set $r_{qq}^{ta} = 0$).

It may be noted here that the origin and departure time of vehicles do not enter in the definition of the splitting proportions. This is because it seems reasonable to assume that, once a vehicle has entered the network, its driver will try to look for the best way to reach his/her destination, irrespective of origin or departure time (this will become important in later sections of the paper). It should also be pointed out that this description of drivers' behavior takes care efficiently of all problems related with "flow-conservation" of vehicles throughout the network. Indeed, if we denote by V_{iq}^t the volume of vehicles headed towards destination q and passing through node i during period t and by v_{aq}^t the number of vehicles headed towards q entering link a during period t, one has

$$v_{aq}^t = r_{iq}^{ta} V_{iq}^t, \qquad i \in N,\ a \in A_i^+,\ q \in Q,\ t \in \{1,\ldots,T\}. \tag{1}$$

2.3 Link and Network Dynamics

To model congestion phenomena on road networks, several authors have followed on the steps of Merchant and Nemhauser (1978) and used the concept of *link exit function*. In their simpler form, exit functions specify the number of vehicles exiting from a link in a given period as a function of the number of vehicles present on the link at the beginning of the period, thus aocounting for the reduction in average speed which is experienced when vehicle density increases. The use of such exit function presents, however, several disadvantages which may easily lead to an unrealistic description of traffic dynamics, even at a macroscopic level.

The first of these disadvantages is that, in some situations, vehicles may actually spend less time on a link than is specified by the "free-flow" (uncongested) traversal time of the link. To solve this problem, we adopt in this paper an approach similar to the one proposed by Drissi and Hameda (1992), which consists in dividing the total time spent on a link a into two parts:

- a *fixed travel time* given by the free-flow time τ_a of a, after which the vehicle joins an exit queue at the end of the link, and
- the *waiting time* spent in the exit queue of the link, which results from the application of the link exit function to the queued vehicles.

To avoid having vehicles moving through several nodes during the same period, thus creating inconsistencies with respect to exit queues, we assume that $\tau_a \geq 1$ for all links. We do not require τ_a to be integer.

Another criticism that has often been leveled against link exit functions is that they do not account for the congestion which may exist at the road junction corresponding to the head-node of a link or on downstream links leading from that junction. In this paper, we use *generalized exit functions* which may include in their definition extensive information about the current state of traffic over the network as well as detailed modelling of the head-node junction. We also allow exit functions to vary with time to account for changes in operating characteristics of links and junctions such as reassignments of lanes, modification of signal settings, etc. If we use the following notation:

e_a^t, exit function of link a in period t,

x_a^t, number of vehicles present on link a at the beginning of period t,

w_a^t, number of vehicles ready to exit from link a during period t,

with $a \in A$ and $t \in \{1, \ldots, T\}$, e_a^t can admit as arguments, apart from the obvious w_a^t, variables such as $x_a^{t'}$, for a small range of $t' \leq t$, to model lower speeds at the head-node caused by high vehicle density in previous periods, $x_{a'}^t$, $a' \in A_{j_a}^+$, to account for the

congestion on downstream links, and $w^t_{a'}$, $a' \in A^-_{j_a}$, to represent congestion at the head-node j_a. The sole restriction that we impose, for the time being, on the exit functions is that they must be continuous in all their arguments (a fairly reasonable assumption in this context).

A third major shortcoming of link exit functions is that they do not preserve the "natural" "first-in first-out" (FIFO) order of vehicles on links, i.e., they may allow vehicles entering a link a in some period t to exit it before vehicles that were already present on a at the beginning of t. To remedy this unsatisfactory situation, we "tag" vehicles (in fact, groups of vehicles) with their entrance time on their current link and force them to exit in accordance with the FIFO rule. To better explain how such a scheme works, we must first define some additional notation:

u^t_a, total number of vehicles exiting link a during period t,

$u^{t't}_a$, total number of vehicles having entered link a during period t' and exiting during t,

u^t_{aq}, number of vehicles headed towards destination q exiting link a during period t,

$u^{t't}_{aq}$, number of vehicles headed towards destination q having entered link a during period t' and exiting during t,

$w^{t't}_a$, total number of vehicles in the exit queue of link a at period t and having entered the link in period t',

$w^{t't}_{aq}$, number of vehicles headed towards destination q in the exit queue of link a at period t and having entered the link in period t',

where $a \in A$, $q \in Q$, $t \in \{1,\ldots,T\}$, $t' \in \{1,\ldots,t-1\}$. We then have:

$$w^{t't}_a := \sum_{q \in Q} w^{t't}_{aq}, \qquad a \in A,\ t \in \{1,\ldots,T\},\ t' \leq t, \tag{2}$$

and

$$w_a^t = \sum_{t'<t} w_a^{t't}, \qquad a \in A,\ t \in \{1,\ldots,T\}. \tag{3}$$

The total number of vehicles exiting during period t is given by the exit function:

$$u_a^t = e_a^t\ (w_a^t,\ \textit{other arguments}), \qquad a \in A,\ t \in \{1,\ldots,T\}. \tag{4}$$

(Note that e_a^t should be defined so that $u_a^t \le w_a^t$).

To enforce the FIFO rule, the groups of vehicles in the exit queue are considered in increasing order of entrance time t':

$$u_a^{t't} = \min\,[w_a^{t't},\ \min(0,\ u_a^t - \sum_{t''<t'} u_a^{t''t})], \qquad a \in A,\ t \in \{1,\ldots,T\},\ t' < t. \tag{5}$$

Note that in Equation (5) the second term in the outer minimization is simply the number of vehicles remaining to be selected for exit after considering all vehicles having entered link a before period t'.

For any given period t', we must now apportion the exiting vehicles amongst destinations. We assume that there is no discrimination between destinations and thus select vehicles in proportion with the size of the groups in the queue, i.e.,

$$u_{aq}^{t't} = \begin{cases} 0, & \text{if } w_a^{t't} = 0; \\ u_a^{t't}(w_{aq}^{t't}/w_a^{t't}), & \text{otherwise,} \end{cases} \qquad a \in A,\ q \in Q,\ t \in \{1,\ldots,T\},\ t' < t. \tag{6}$$

Finally, we can compute the total number of vehicles headed for each destination exiting:

$$u_{aq}^t = \sum_{t'<t} u_{aq}^{t't}, \qquad a \in A,\ q \in Q,\ t \in \{1,\ldots,T\}. \tag{7}$$

To complete our description of link dynamics, we need some equations to compute the V_{iq}^t's of section 2.2, the x_a^t's and the $w_{aq}^{t't}$'s.

The number of vehicles passing through a node in a period is the number of exiting vehicles on its backward-star links to which must be added vehicles entering the network if this node is an origin:

$$V_{iq}^t = \begin{cases} \sum_{a \in A_i^-} u_{aq}^t, & i \in N \setminus O, \\ g_{iq}^t + \sum_{a \in A_i^-} u_{aq}^t, & i \in O, \end{cases} \qquad q \in Q,\ t \in \{1,\ldots,T\}. \tag{8}$$

For the total number of vehicles present on a link at the beginning of a period, we have a simple flow-accounting equation:

$$x_a^t = x_a^{t-1} + \sum_{q \in Q} v_{aq}^{t-1} - u_a^{t-1}, \qquad a \in A,\ t \in \{2, \ldots, T\}, \tag{9}$$

with $x_a^1 \equiv 0,\ a \in A$.

Let us now turn to the determination of the number of vehicles in the exit queue. If the free-flow time τ_a of link a is an integer, the full group of vehicles entering the link will join the exit queue τ_a periods later. If, to the contrary, we allow τ_a to be a real number, we face a problem: when will the vehicles join the queue? If we assume the entrance times of vehicles on link a to be uniformly spread out over any period t, then a fraction $(\lfloor \tau_a \rfloor + 1 - \tau_a)$ of these vehicles will complete the traversal during period $(t + \lfloor \tau_a \rfloor)$ and the remainder (a proportion $(\tau_a - \lfloor \tau_a \rfloor)$)will do so during the next period (note that this division of vehicles between the periods effectively produces an average traversal time of τ_a periods). We thus obtain:

$$w_{aq}^{t't} = \begin{cases} w_{aq}^{t'(t-1)} - u_{aq}^{t'(t-1)}, & \text{if } t' < t - \lfloor \tau_a \rfloor - 1, \\ w_{aq}^{t'(t-1)} - u_{aq}^{t'(t-1)} + (\tau_a - \lfloor \tau_a \rfloor) v_{aq}^{t'}, & \text{if } t' = t - \lfloor \tau_a \rfloor - 1, \\ (\lfloor \tau_a \rfloor + 1 - \tau_a) v_{aq}^{t'}, & \text{if } t' = t - \lfloor \tau_a \rfloor, \\ 0, & \text{if } t' > t - \lfloor \tau_a \rfloor, \end{cases} \tag{10}$$

with $a \in A,\ q \in Q,\ t \in \{1, \ldots, T\}$.

This completes the description of traffic dynamics over the network.

2.4 Assigning Flows on the Network

We now present a "conceptual" dynamic assignment procedure using the equations previously defined. Our purpose here is not to propose a computationally efficient algorithm, but simply to show how these equations interact to produce a valid assignment under the assumptions made at the beginning of this section.

The procedure proceeds sequentially from period 1 to period T. In each period t, the sequence of operations is the following:

(1) Determine the number of vehicles present on each link at the beginning of the period $\{x_a^t\}$ using Equation (9).

(2) Update the exit queues $\{w_{aq}^{t't}\}$ using Equation (10) and the totals $\{w_a^{t't}\}$ and $\{w_a^t\}$ with Equations (2) and (3).

(3) Determine the number of vehicles exiting each link $\{u_a^t\}$ by applying the exit functions (Equation (4)), enforce the FIFO rule (Equation (5)) and select the exiting groups $\{u_{aq}^{t't}\}$ according to Equation (6).

(4) Compute the number of vehicles moving through each node during the period $\{V_{iq}^t\}$ using Equations (7) and (8) and distribute them in accordance with the drivers' strategies $\{r_{iq}^{ta}\}$ (Equation (1)).

As the assignment is performed, one may also compute the proportion of the vehicles entering a link a during a given period t which exit during any later period t'; this is denoted $p_a^{tt'}(\underline{r})$ to emphasize the dependency of these proportions on drivers' strategies (hence the notation $\underline{r}$). One has

$$p_a^{tt'}(\underline{r}) = u_a^{tt'} \Big/ \sum_{q \in Q} v_{aq}^t, \qquad a \in A,\ t \in \{1, \ldots, T\},\ t' > t, \tag{11}$$

for all combinations (a, t) such that $\sum_{q \in Q} v_{aq}^t \neq 0$. When $\sum_{q \in Q} v_{aq}^t = 0$, $p_a^{tt'}(\underline{r})$ should be undefined, but it would still be interesting to know when a vehicle entering link a at t would have exited it. We can distinguish two cases:

- the first one corresponds to the situation where link a is uncongested at time t, i.e., when all vehicles entering link a before t exit it no later than in period $(t + \lfloor \tau_a \rfloor)$; when this occurs the vehicle would not have suffered any delay and the time on the link should be the free-flow time; the discussion immediately

preceding Equation (10) thus applies and we may write:

$$p_a^{tt'}(\underline{r}) = \begin{cases} \lfloor \tau_a \rfloor + 1 - \tau_a, & \text{if } t' = t + \lfloor \tau_a \rfloor, \\ \tau_a - \lfloor \tau_a \rfloor, & \text{if } t' = t + \lfloor \tau_a \rfloor + 1, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

- the second case is the congested one in which some vehicles preceding our hypothetical vehicle exit later than $(t+\tau_a)$; in this situation this hypothetical vehicle would exit immediately after the last vehicle preceding it in the time period t^* given by $max\ \{t' \mid u_a^{t''t'} > 0,\ t'' < t\}$, and we thus obtain

$$p_a^{tt'}(\underline{r}) = \begin{cases} 1, & \text{if } t' = t^*, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

It must be pointed out that the $p_a^{t't}(\underline{r})$'s apply homogeneously to the vehicles headed towards all destinations. This can be ascertained by verifying that, if $v_{aq}^t > 0$, then

$$p_a^{tt'}(\underline{r}) = u_{aq}^{tt'} / v_{aq}^t$$

(this stems from the proportionality assumption underlying Equation (6)).

2.5 Computing Average Travel Times

Within the framework that we have defined, we can easily compute the average travel times induced by the drivers' strategies between any node in the network and any destination at any point in time. As we shall see, the proportions $p_a^{tt'}(\underline{r})$ will play an important role in these computations. Indeed, if we define .

$\pi_{iq}^{ta}(\underline{r})$, the average travel time from node i to destination q if link a is entered in period t,

$\pi_{iq}^{t}(\underline{r})$, the average travel time from node i to destination q if a vehicle passes through or departs node i in period t,

$\pi_{iq}^{t*}(\underline{r})$, the "best" average time from node i to destination q in period t,

with $i \in N$, $q \in Q$, $a \in A_i^+$ and $t \in \{1, \ldots, T\}$, we have the following relationships:

$$\pi_{iq}^{ta}(\underline{r}) = \sum_{t'>t} p_a^{tt'}(\underline{r})\left[(t'-t) + \pi_{j_a q}^{t'}(\underline{r})\right], \tag{14}$$

$$\pi_{iq}^{t}(\underline{r}) = \begin{cases} 0, & \text{if } i = q, \\ \sum_{a \in A_i^+} r_{iq}^{ta} \pi_{iq}^{ta}(\underline{r}), & \text{if } i \neq q, \end{cases} \tag{15}$$

$$\pi_{iq}^{t*}(\underline{r}) = \begin{cases} 0, & \text{if } i = q, \\ \min_{a \in A_i^+}\left[\pi_{iq}^{ta}(\underline{r})\right], & \text{if } i \neq q. \end{cases} \tag{16}$$

Several remarks are now in order:

1) The π values have an "$\underline{r}$" argument to stress the fact that average travel times are highly dependent upon the strategies chosen by drivers.

2) In Equation (14), the term $(t' - t)$ corresponds to the time spent on link a, while $\pi_{j_a q}^{t'}(\underline{r})$ represents the average travel incurred to reach q after exiting link a during period t'.

3) In (14), if one cannot be guaranteed to exit link a before the end of period T (this would occur, for instance, if $t + \tau_a > T$), we set $\pi_{iq}^{ta}(\underline{r})$ equal to $+\infty$ by convention.

4) $\pi_{iq}^{t*}(\underline{r})$ does not correspond to a dynamic shortest-path from node i to destination q at period t, but rather to the best link choice out of a given current traffic conditions if strategy $\underline{r}$ is applied after exiting a. Those values will be crucial to the definition of day-to-day dynamics in section 4.

5) Once the traffic assignment procedure has been executed and the $p_a^{tt'}(\underline{r})$'s determined, the π values can be easily computed using a backward recursion from period T to period 1, since the average travel times for period t depend solely on travel times in later periods.

3 DEPARTURE TIMES DETERMINATION

In the previous section, we made the assumption that the departure times for all vehicles were fixed, but in most networks this assumption is not realistic and the choice of departure time is indeed one of the most important decisions that drivers have to make. Obviously, changes in departure times are part of the day-to-day adjustments that drivers make to their travel strategy and it is clearly not possible to represent such changes in a model covering within-day traffic assignment. The purpose of this section is thus not to address directly these changes, but rather to indicate how the key elements relevant for these decisions can be imbedded in our model.

3.1 Arrival and Departure Time Preferences

In the context of dynamic traffic assignment, it is generally considered that one of the critical factors in the determination of departure time decisions is the fact that drivers have strong preferences regarding the time at which they travel. In the past, several authors have tried to model these preferences using penalties for early and late arrival at the destination ("early" and "late" being defined with respect to some "ideal" arrival time at the destination). We believe that such an approach is incorrect in the sense that once drivers have departed their origin, they would rather arrive at their destination as early as possible. Furthermore, the introduction of penalties for early arrival in a model may lead to awkward solutions where drivers deliberately lose time over the network to arrive at the "right" time at their destination. To balance the desire for arriving as early as possible one must take into account the preference of drivers to leave as late as possible. In this paper, we thus consider three components for the total travel cost: the travel time itself, a penalty related to the departure time and one for the arrival time (these penalties being expressed in equivalent travel time units).

Since in our model, we distinguish vehicles with respect to their destination, we are forced to assume that all drivers headed towards a given destination share the same pref-

erences (i.e., penalties) regarding arrival time. Should this not be case, destinations must be replicated and drivers divided in accordance with their preferences. We denote by $\overline{S}_q(t)$ the penalty for arriving at destination q at time t. Having assumed that drivers prefer to arrive as early as possible, $\overline{S}_q(t)$ should be a monotone increasing function of t for each $q \in Q$. Arrival time penalties can easily be accounted for when computing the average travel time to the destination. If in Equations (15) and (16), we set $\pi^t_{qq}(\underline{r})$ and $\pi^{t*}_{qq}(\underline{r})$ equal to $\overline{S}_q(t)$ instead of 0, all π values now represent "average travel time plus arrival time penalty" without any additional computational effort.

Departure time preferences can be handled in more detail, since there is nothing which prevents us from defining penalties $\tilde{S}_{pq}(t)$ for each specific O-D pair (p,q). This might be of interest since it allows us to model the fact that drivers whose origin is far away from their destination probably accept more readily early departure times. Similarly to the situation we were facing when dealing with arrival penalties, we may be forced to replicate origins if drivers' preferences are too dissimilar for some O-D pairs. The $\tilde{S}_{pq}(t)$ functions are assumed to be monotone decreasing in t to represent the preference for late departures. The total travel cost for a vehicle leaving origin p for destination q at time t given drivers' strategy vector $\underline{r}$, denoted $\tilde{\pi}^t_{pq}(\underline{r})$, is then given by

$$\tilde{\pi}^t_{pq}(\underline{r}) = \tilde{S}_{pq}(t) + \pi^t_{pq}(\underline{r}), \tag{17}$$

for all $p \in P$, $q \in Q$ and $t \in \{1,\ldots,T\}$.

3.2 Departure Times and Drivers' Strategies

At the beginning of this section, it was mentioned that the choice of a departure time is one of the most important decisions that a driver must make. It would thus seem natural to make this decision part of his/her travel strategy. If we assume for every O-D pair (p,q) a fixed demand g_{pq} over the study horizon, this can easily be accomodated within the framework of section 2, in which we consider aggregate driver strategies for groups of vehicles. If we define for $p \in P$, $q \in Q$, $t \in \{1,\ldots,T\}$,

r^t_{pq}, the proportion of vehicles from O-D pair (p,q) departing during period t, then g^t_{pq} becomes a variable of the traffic assignment model and is given by

$$g^t_{pq} = r^t_{pq} g_{pq}, \qquad p \in P,\ q \in Q,\ t \in \{1,\ldots,T\}. \tag{18}$$

If we consider the r^t_{pq} values to be part of the drivers' strategy vector $\underline{r}$, we can then determine the average total travel cost from p to q with respect to $\underline{r}$ as

$$\tilde{\pi}_{pq}(\underline{r}) = \sum_t r^t_{pq} \tilde{\pi}^t_{pq}(\underline{r}), \qquad p \in P,\ q \in Q. \tag{19}$$

We can also define the best average total travel cost for (p,q) under $\underline{r}$ as

$$\tilde{\pi}^*_{pq}(\underline{r}) = \min_t \left[\tilde{\pi}^t_{pq}(\underline{r})\right], \qquad p \in P,\ q \in Q. \tag{20}$$

4 DAY TO DAY DYNAMICS

In sections 2 and 3, we have developed at length a dynamic traffic assignment model which allows us to determine the traffic patterns and associated travel costs which result from the simultaneous application of aggregate drivers' travel strategies described by a strategy $\underline{r}$. In this model, no provision was made for any optimizing behavior on the part of drivers with respect to their travel costs. We now address this question.

4.1 Behavioral Assumption

We assume that drivers are rational and strive to minimize their own travel costs. Furthermore, we assume that they have some knowledge of the network over which they travel, knowledge which can be used to try to improve their situation. More precisely, we make the assumption that when deciding upon their travel strategy for a given day $(d+1)$, drivers have available the information on traffic conditions on the previous day d. If we denote by $\underline{r}(d)$ the strategy vector used on day d, this information is summarized in vectors $\underline{\pi}(\underline{r}(d))$ and $\tilde{\pi}(\underline{r}(d))$. The strategy vector for day $(d+1)$ is then given by a *strategy*

adjustment mapping $\Phi(\underline{r}(d), \underline{\pi}(\underline{r}(d)), \underline{\tilde{\pi}}(\underline{r}(d))$ which we assume to be continuous in all its parameters. Furthermore, Φ should provide for some improvement in the decision-making for the cost vectors considered. Since in our framework decisions are defined locally (at specific nodes and time periods), these improvements should be local in nature. We thus require Φ to produce a strategy vector $\underline{r}(d+1)$ such that:

- for each node-destination-period triplet (i,q,t) with $\pi^t_{iq}(\underline{r}(d)) > \pi^{t*}_{iq}(\underline{r}(d))$,

$$\sum_{a \in A_i^+} r^{ta}_{iq}(d+1)\pi^{ta}_{iq}(\underline{r}(d)) < \pi^t_{iq}(\underline{r}(d)) = \sum_{a \in A_i^+} r^{ta}_{iq}(d)\pi^{ta}_{iq}(\underline{r}(d)),$$

- for each O-D pair (p,q) with $\tilde{\pi}_{pq}(\underline{r}(d)) > \tilde{\pi}^*_{pq}(\underline{r}(d))$,

$$\sum_t r^t_{pq}(d+1)\tilde{\pi}^t_{pq}(\underline{r}(d)) < \tilde{\pi}^*_{pq}(\underline{r}(d)) = \sum_t r^t_{pq}(d)\tilde{\pi}^t_{pq}(\underline{r}(d)).$$

The first condition corresponds to situations where the distribution of vehicles on outgoing links can be improved, while the second covers possible improvements in departure time decisions. In the remaining situations, decisions are not modified.

Given the current state of research on drivers' behavior, it is not possible to specify more precisely these strategy adjustment mappings and, in consequence, to study transient variations in traffic patterns. We can however define an equilibrium state for the day-to-day dynamics.

4.2 A Dynamic Equilibrium Condition

Given the restrictions on Φ, it is easily seen that a strategy vector $\underline{r}$ is not modified by Φ if and only

$$\begin{cases} \pi^t_{iq}(\underline{r}) = \pi^{t*}_{iq}(\underline{r}), & i \in N,\ q \in Q,\ t \in \{1,\ldots,T\}, \\ \tilde{\pi}_{pq}(\underline{r}) = \tilde{\pi}^*_{pq}(\underline{r}), & p \in P,\ q \in Q. \end{cases}$$

This can be rewritten in the equivalent form:

$$\begin{cases} \displaystyle\sum_{a \in A_i^+} r^{ta}_{iq}\pi^{ta}_{iq}(\underline{r}) = \min_{a \in A_i^+}\left[\pi^{ta}_{iq}(\underline{r})\right], & q \in Q,\ i \in N \setminus \{q\},\ t \in \{1,\ldots,T\}, \\ \displaystyle\sum_t r^t_{pq}\tilde{\pi}^t_{pq}(\underline{r}) = \min_t\left[\tilde{\pi}^t_{pq}(\underline{r})\right], & p \in P,\ q \in Q. \end{cases}$$

It should be pointed out that, since every strategy vector $\underline{r}$ induces through the "within-day" assignment model a unique traffic pattern, the previous equilibrium condition could fairly easily be stated in terms of the traffic variables themselves, if desired.

4.3 Existence of an Equilibrium

We would now like to determine whether there exists equilibrium strategy vectors. This question can easily be settled under the assumptions made earlier. First recall that the link exit functions were assumed to be continuous in all their arguments. This implies that, given the link and network dynamics defined in section 2, the average cost vectors $\underline{\pi}(\underline{r})$ and $\underline{\tilde{\pi}}(\underline{r})$ are continuous mappings with respect to $\underline{r}$. Since the strategy adjustment mapping Φ was also assumed to be continuous in $\underline{r}$, $\underline{\pi}(\underline{r})$ and $\underline{\tilde{\pi}}(\underline{r})$, it follows, when Φ is interpreted as a mapping from the set of feasible strategy vectors (denoted $\mathcal{R}$) onto itself, that Φ is a continuous mapping from $\mathcal{R}$ to $\mathcal{R}$. Now, $\mathcal{R}$ being the Cartesian product of a finite number of $[0,1]$ intervals, it is compact and the Fixed-point Theorem guarantees the existence of at least one fixed-point of Φ in $\mathcal{R}$, i.e., an equilibrium strategy vector.

5 CONCLUSION

We have described a discrete-time Dynamic Traffic Assignment model for general networks with multiple origin-destination pairs and various preference patterns with regards to departure and arrival times. The key features of this model are:

- detailed modelling, at the macroscopic level, of interactions at road and street junctions;
- strict enforcement of "first-in first-out" rules on all links at all times;
- consistent endogeneous average travel costs from any node to any destination in all periods in a discrete-time setting;
- an efficient link-node description of traffic flows and travel decisions.

The model was shown to lead to a natural Dynamic Traffic Equilibrium condition which, under reasonable continuity assumptions, is satisfied by at least one driver travel strategy vector, or equivalently, one traffic pattern.

The complex question of the uniqueness of the equilibrium strategy was not studied in this paper; it certainly constitutes an interesting topic for further research in this area. In general, one would not expect networks to exhibit that property, but it may be possible to derive a set of necessary conditions relative to network parameters to guarantee uniqueness.

When we started developing this model, our objective was to solve real-life traffic assignment problems, i.e., problems defined on fairly large networks with several O-D pairs. Even though our model has not yet been implemented, we believe that this aim will soon be achieved. We are now working on the design of efficient computational procedures to perform both the "within-day" assignment and the day-to-day strategy adjustment and we should be able to report computational results in the near future.

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